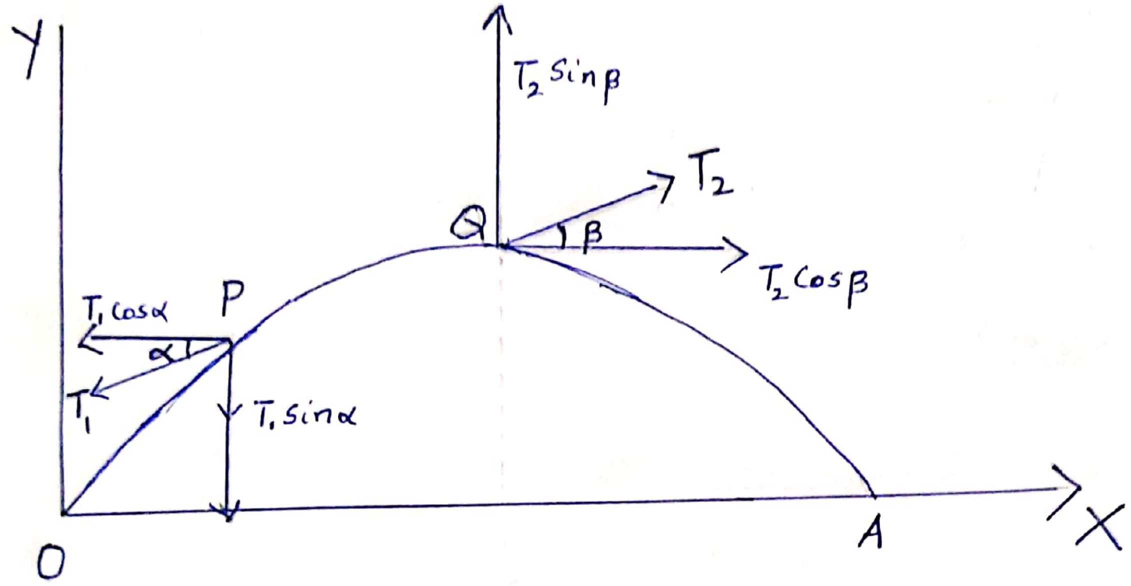
Module - 2

ONE DIMENSIONAL WAVE EQUATION

DERIVATION



Consider an elastic string strecthed to its length l & allingned or placed along the X-axis with its two ends $x=0$ & $x=L$ fixed. Let ρ be the mass of the string per unit length. Let the function $u(x,t)$ denote the displacement of the string at any point x & at any time $t>0$ from the equilibrium position. when the string is distorted then it vibrates.

Let T_1 & T_2 be the tensions at the end points P & Q of the string. Since the points of the string move vertically there is no motion in the horizontal direction. Thus the sum of the forces in the horizontal direction must be zero

$$-T_1 \cos\alpha + T_2 \cos\beta = 0$$

$$T_1 \cos\alpha = T_2 \cos\beta = T, \quad \text{a constant} \quad \text{--- (1)}$$

In the vertical direction, there are two forces namely the vertical components $-T_1 \sin\alpha$ & $T_2 \sin\beta$.

Hence by Newton's second law of motion.

Resultant forces = mass x acceleration.

ie $$-T_1 \sin\alpha + T_2 \sin\beta = \rho \, \Delta x \frac{\partial^2 u}{\partial t^2} \quad \text{--- (2)}$$

(2) ÷ (1) $$\Rightarrow -\frac{T_1 \sin\alpha}{T_1 \cos\alpha} + \frac{T_2 \sin\beta}{T_2 \cos\beta} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$-\tan\alpha + \tan\beta = \rho \frac{\Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\tan\beta - \tan\alpha}{\Delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} \quad \text{--- (3)}$$

Now $\tan\alpha$ & $\tan\beta$ are the slopes of the string at the pt points with co-ordinates x & $x+\Delta x$

So that $$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_{\text{at } x}.$$

$$\tan\beta = \left(\frac{\partial u}{\partial x}\right)_{\text{at } x+\Delta x}.$$

$\therefore$ (3) $$\Rightarrow \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

As $\Delta x \to 0$ $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2}.$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{\rho}$$

This equation is called one dimensional wave equation.

Important assumptions of one dimensional wave eqn.

→ Motion takes place entirely in X-Y plane and each particle of the string moves ~~p~~ perpendicular to the equillibrium position OA of the string.

→ The string is perfectly flexible and does not offer ~~to bending~~ resistance to bending.

→ Tension in the string is so large so that forces due to the weight of the string can be neglected.

SOLUTIONS OF WAVE EQUATION

By method of Separation of variables

One dimensional wave equation is

$$U_{xx} = \frac{1}{c^2} U_{tt} \quad \text{—} \ (1)$$

Let $U = XT$ be the solution.

$$U_x = TX' \qquad U_{xx} = TX''$$

$$U_t = XT' \qquad U_{tt} = XT''$$

Substituting we get

$$TX'' = \frac{1}{c^2} XT''$$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T}$$

$$\frac{X''}{X} = k \implies X'' - Xk \implies X'' - Xk = 0$$

$$\implies \cancel{X''} \ (D^2 - k)X = 0$$

$\therefore$ The auxiliary eqn is $m^2 - k = 0$

$$m = \pm\sqrt{k}$$

Let $\frac{1}{c^2}\frac{T''}{T} = k$

$$\frac{T''}{T} = kc^2 \Rightarrow T'' - kc^2 T = 0.$$

$\Rightarrow (D^2 - kc^2) = 0$ Aux. eqn is $(m^2 - kc^2)\cancel{X} = 0.$

$$m^2 = kc^2 \Rightarrow m = \pm\sqrt{k}\,c.$$

Case - I when $k = 0$

$$m = 0, 0.$$

$$X = (C_1 + C_2 x)e^{0x}$$

$$X = C_1 + C_2 x$$

$$m = 0, 0$$

$$T = (C_3 + C_4 t)e^{0t}$$

$$T = C_3 + C_4 t$$

Case II when k is -ve. Say $k = -p^2$

$$m = \pm pi$$

$$X = e^{0x}(C_1 \cos px + C_2 \sin px)$$

$$X = C_1 \cos px + C_2 \sin px.$$

$$m = \pm pci$$

$$T = e^{0t}(C_3 \cos pct + C_4 \sin pct)$$

$$T = C_3 \cos pct + C_4 \sin pct$$

Case III when k is +ve say $k = p^2$

$m = \pm p$	$m = \pm pc$
$X = C_1 e^{px} + C_2 e^{-px}$	$T = C_3 e^{pct} + C_4 e^{-pct}$

$$U = (C_1 + C_2 x)(C_3 + C_4 t)$$

$$U = (C_1 \cos px + C_2 \sin px)(C_3 \cos pct + C_4 \sin pct)$$

$$U = (C_1 e^{px} + C_2 e^{-px})(C_3 e^{pct} + C_4 e^{-pct})$$

But we are dealing with problems on vibration $\therefore$ Solution must include trignometric functions $\therefore$ the Only possible Solution is ~~$U = (C_1 \cos px + C_2 \sin px)$~~

$$U = (C_1 \cos px + C_2 \sin px)(C_3 \cos pct + C_4 \sin pct)$$

D-Alembert's solution of wave equation

The solution of the vibration in an infinitely long elastic string is known as D-Alembert's solution. Hence for infinitely long string, the wave equation is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $-\infty < x < \infty$, $t > 0$ ⟶ ①

Let $p = x - ct$ & $q = x + ct$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot 1 + \frac{\partial u}{\partial q}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial p}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial q}\right)$$

$$= \frac{\partial}{\partial p}\left(\frac{\partial u}{\partial p}\right) \cdot \frac{\partial p}{\partial x} + \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial p}\right) \cdot \frac{\partial q}{\partial x}$$

$$+ \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial q}\right) \frac{\partial q}{\partial x} + \frac{\partial}{\partial p}\left(\frac{\partial u}{\partial q}\right) \frac{\partial p}{\partial x}.$$

$$= \frac{\partial^2 u}{\partial p^2} \cdot \frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial p \partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial^2 u}{\partial q^2} \cdot \frac{\partial q}{\partial x}$$

$$+ \frac{\partial^2 u}{\partial p \partial q} \cdot \frac{\partial p}{\partial x}$$

$$= \frac{\partial^2 u}{\partial p^2} + \frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2} + \frac{\partial^2 u}{\partial p \partial q}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial p^2} + 2\frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2} \quad \text{--- (2)}$$

Now $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial t} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial t}$

$$= \frac{\partial u}{\partial p}(-c) + \frac{\partial u}{\partial q} c$$

$$= c\left[-\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right]$$

$$\frac{\partial^2 u}{\partial t^2} = c\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial q}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial p}\right)\right]$$

$$= c\left\{\left[\frac{\partial}{\partial p}\left(\frac{\partial u}{\partial q}\right) \cdot \frac{\partial p}{\partial t} + \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial q}\right)\frac{\partial q}{\partial t}\right] - \frac{\partial}{\partial p}\left(\frac{\partial u}{\partial p}\right) \cdot \frac{\partial p}{\partial t} - \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial p}\right) \cdot \frac{\partial q}{\partial t}\right\}$$

$$= c\left[\frac{\partial^2 u}{\partial p \partial q}(-c) + \frac{\partial^2 u}{\partial q^2}\cdot c + \frac{\partial^2 u}{\partial p^2}\cdot c - \frac{\partial^2 u}{\partial p \partial q}\cdot c\right]$$

$$= c^2\left[\frac{\partial^2 u}{\partial p^2} + \frac{\partial^2 u}{\partial q^2} - 2\frac{\partial^2 u}{\partial p \partial q}\right] \quad \text{---} (3)$$

Substituting (2) & (3) in (1).

$$c^2\left[\frac{\partial^2 u}{\partial p^2} - 2\frac{\partial^2 u}{\partial p \partial q} + \frac{\partial^2 u}{\partial q^2}\right] = c^2\left[\frac{\partial^2 u}{\partial p^2} + \frac{\partial^2 u}{\partial q^2} + 2\frac{\partial^2 u}{\partial p \partial q}\right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial p \partial q} = 0 \quad \text{---} (4)$$

Integrating (4) w.r.t p.

$$\frac{\partial u}{\partial q} = h(q)$$

Integrating w.r.t q

$$u = g(q) + f(p)$$

ie $u(p,q) = g(x+ct) + f(x-ct)$

This is the D'- Alembert's solution of wave equation.

Note: The boundary conditions which the wave equation has to ~~sta~~ satisfy are

$u = 0$, when $x = 0$, ie $\boxed{u(0,t) = 0}$

$u = 0$ when $x = L$ ie $\boxed{u(L,t) = 0}$

If the string is made to vibrate by pulling it into a curve $u = f(x)$ & then releasing it

The initial conditions are

$$u = f(x) \quad \text{when } t = 0$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{when } t = 0.$$

ie
$$\boxed{\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t} &= 0 \quad \text{when } t = 0 \end{aligned}}$$

1. ~~Finding~~ one ~~dimensional~~ ~~wave equation~~ ~~such~~ ~~that~~ ~~initial~~ ~~velocity~~ ~~zero~~ ~~and~~

1. Find the displacement of a finite string of length L that is fixed at both ends and is released from rest with an initial displacement $f(x)$.

One dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c}\frac{\partial^2 u}{\partial t^2}$$

Solution is given by

$$u(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pct + C_4 \sin pct) \quad (1)$$

The boundary conditions are.

$u(0,t)=0$

$u(L,t)=0$

Intial Conditions are.

$\frac{\partial u}{\partial t} = 0$ when $t=0$

$u(x,0) = f(x)$

Applying the boundary condition $u(0,t)=0$ in eqn (1) we have.

$$0 = C_1 [C_3 \cos pct + C_4 \sin pct]$$

$$\Rightarrow \boxed{C_1 = 0}$$

$$u(x,t) = C_2 \sin px [C_3 \cos pct + C_4 \sin pct] \quad (2)$$

Applying the 2nd boundary condition $u(L,t)=0$ in (2)

$$0 = C_2 \sin pL \left[C_3 \cos pct + C_4 \sin pct \right]$$

$$\Rightarrow \sin pL = 0 \Rightarrow pL = n\pi$$

i.e. $\boxed{p = \frac{n\pi}{L}}$

$$u(x,t) = C_2 \sin \frac{n\pi x}{L} \left[C_3 \cos \frac{n\pi ct}{L} + C_4 \sin \frac{n\pi ct}{L} \right]$$

$$u(x,t) = \left[a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}$$

$a_n = C_2 C_3$

$b_n = C_2 C_4$

~~Applying the initial condition~~

~~$\frac{\partial u}{\partial t} = 0$ when $t = 0$~~

~~$\frac{\partial u}{\partial t} =$~~

Now adding up the solutions for different values of n we get

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L} \quad (3)$$

Applying the initial condition. $\frac{\partial u}{\partial t} = 0$ when $t = 0$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[a_n - \sin \frac{n\pi ct}{L} \times \frac{n\pi c}{L} + b_n \cos \frac{n\pi ct}{L} \times \frac{n\pi c}{L} \right] \sin \frac{n\pi x}{L}$$

$$0 = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L} \Rightarrow \boxed{b_n = 0}$$

$$(3) \Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi ct}{L} \sin \frac{n\pi x}{L} \quad (4)$$

Applying initial condition $u(x,0) = f(x)$

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} = f(x)$$ which is a

fourier sine series where.

$$\boxed{a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.}$$

∴ the solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi c t}{L} \sin \frac{n\pi x}{L}$$

where.

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

2. A tightly-streched string of length L is fixed at both ends. Find the displacement $u(x,t)$ if the string is given an initial displacement $f(x)$ and an initial velocity $g(x)$.

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

Boundary conditions $\boxed{\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0. \end{array}}$

Initial conditions $u(x,0) = f(x)$

~~u(~~ $\frac{\partial u}{\partial t}(x,0) = g(x)$

Using method of separation of variables.

$$u(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pct + C_4 \sin pct) \quad (1)$$

Applying the boundary condition $u(0,t)=0$

$$\boxed{C_1 = 0}$$

$$u(x,t) = C_2 \sin px \left[C_3 \cos pct + C_4 \sin pct\right] \quad (2)$$

Applying 2nd condition $u(L,t) = 0$

we have $\boxed{p = \frac{n\pi}{L}}$ $n = 1, 2, 3 \ldots$

$$u(x,t) = \left[a_n \cos \frac{n\pi ct}{L} + \cancel{\sin}\, b_n \sin \frac{n\pi ct}{L}\right] \sin \frac{n\pi x}{L}$$

or $$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L}\right] \sin \frac{n\pi x}{L} \quad (3)$$

Applying 3rd condition $u(x,0) = f(x)$ we have

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} = f(x)$$

where $\boxed{a_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.}$ — (A)

Applying 4th condition $\frac{\partial u}{\partial t} = g(x)$ at $t=0$

or $\frac{\partial u}{\partial t}(x,0)$ we have.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left[-a_n \sin \frac{n\pi ct}{L} + b_n \cos \frac{n\pi ct}{L}\right] \sin \frac{n\pi x}{L}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin \frac{n\pi x}{L}$$

which is also a half range sine series.

where $\frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$

or $\boxed{b_n = \frac{2}{n\pi c}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.}$ — (B)

∴ eqn (3) is the required solution where a_n & b_n are given by the equations (A) & (B).

A uniform string of length 60 cm is subjected to a constant tension of 2 kg. If the ends are fixed and the initial displacement $u(x,0) = 60x - x^2$, $0 < x < 60$, while the initial velocity is zero. Find the displacement function

1-D wave equation is $u_{xx} = \frac{1}{c^2} u_{tt}$

By method of separation of variables the solution is given by

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct) \quad \text{--- (1)}$$

The boundary conditions are

$u(0,t) = 0$ and ~~$u(60$~~ $u(60,t) = 0$

Initial conditions are.

$\frac{\partial u}{\partial t} = 0$ when $t = 0$.

$u(x,0) = f(x) = \cancel{60 - x^2}\ 60x - x^2$

Applying the 1st condition $u(0,t) = 0$

we get $c_1 = 0$

Applying 2nd condition $u(60,t) = 0$.

we have $p = \frac{n\pi}{60}$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi ct}{60} + b_n \sin \frac{n\pi ct}{60} \right] \sin \frac{n\pi x}{60}. \quad \text{--- (2)}$$

Applying the initial condition $\frac{\partial u}{\partial t} = 0$ when $t = 0$

we have $b_n = 0$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi c t}{60} \cdot \sin\frac{n\pi x}{60} \quad (3)$$

Applying the initial condition $u(x,0) = f(x)$

$= 60x - x^2$ we get

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{60}$$

$$\therefore a_n = \frac{2}{60}\int_0^{60} (60x - x^2) \sin\left(\frac{n\pi x}{60}\right) dx.$$

$$= \frac{1}{30}\left[(60x - x^2)\left[\frac{-\cos\frac{n\pi x}{60}}{\frac{n\pi}{60}}\right] - (60-2x)\left(\frac{-\sin\frac{n\pi x}{60}}{\frac{n^2\pi^2}{60^2}}\right) + (-2)\left[\frac{\cos\frac{n\pi x}{60}}{\frac{n^3\pi^3}{60^3}}\right]\right]_0^{60}$$

$$= \frac{1}{30}\left\{\left[0 - 0 + (-2)\frac{60^3 (-1)^n}{n^3\pi^3}\right] - \left[0 - 0 - \frac{2\times 60^3}{n^3\pi^3}\right]\right.$$

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$$= \frac{1}{30}\left[\frac{2\times 60^3}{n^3\pi^3}\left(1-(-1)^n\right)\right]$$

$$= \frac{14400}{n^3\pi^3}\left[1-(-1)^n\right] = \begin{cases} \frac{28800}{n^3\pi^3} & n \text{ is odd} \\ 0. & n \text{ is even} \end{cases}$$

4) A tightly stretched string with fixed end points $x=0$ and $x=l$. is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l-x)$. Find displacement of the string at any distance x from one-end at any time.

1-D wave eqn is $u_{xx} = \frac{1}{c^2} u_{tt}$

By method of separation of variables the solution is given by

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct) \quad \text{—} (1)$$

B.C

$u(0,t) = 0, \quad u(l,t) = 0$

I.C

$u(x,0) = 0$

$\frac{\partial u}{\partial t} = \lambda x(l-x)$ when $t=0$.

Applying ~~B.C.~~ Boundary condition $u(0,t)=0$ ~~$u(x,0)=0$~~

$$u(0,t)=0 \implies c_1 = 0$$

$$u(l,t)=0 \implies p = \frac{n\pi}{l}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi c t}{l} + b_n \sin\frac{n\pi c t}{l}\right]\sin\frac{n\pi x}{l} \quad \text{(2)}$$

Applying the initial condition $u(x,0)=0$

we have $a_n = 0$.

Applying ~~the~~ $\frac{\partial u}{\partial t} = 0$ when $t=0$

we have

$$b_n = \frac{2}{n\pi c}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{n\pi c}\left\{(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right\}_0^l$$

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$$= \frac{4\lambda l^3}{n^3 \pi^3 c}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} 0 & n \text{ is even.} \\ \dfrac{8\lambda l^3}{n^4 \pi^4 c} & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$ is

the solution where.

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even.} \\ \dfrac{8\lambda l^3}{n^4 \pi^4 c} & \text{if } n \text{ is odd.} \end{cases}$$

5.) A tightly streched string of length a. with fixed ends is initially in equilibrium position. Find displacement $u(x,t)$ of the. string if it is set vibrating by giving each. of its points a velocity $V_0 \sin\left(\frac{\pi x}{a}\right)$

1-D wave eqn is $u_{xx} = \frac{1}{c^2} u_{tt}$

By method of Separation of variables.

its solution is given by

$$u(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pct + C_4 \sin pct) \quad \text{--- (1)}$$

<u>B.C</u>

$u(0,t) = 0 \qquad u(a,t) = 0$

I.C

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t} = V_0 \sin\left(\frac{\pi x}{a}\right) \quad \text{when } t=0$$

Applying $u(0,t)=0$ we have $C_1=0$

$u(l,t)=0$ we have $p = \frac{n\pi}{a}$

~~$u(x,0)=0$~~

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi ct}{a} + b_n \sin\frac{n\pi ct}{a}\right]\sin\left(\frac{n\pi x}{a}\right) \quad (2)$$

Applying $u(x,0)=0$ we have $a_n = 0$.

$$\frac{\partial u}{\partial t} = V_0 \sin\left(\frac{\pi x}{a}\right) \quad \text{when } t=0$$

~~we~~ have

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi ct}{a}\sin\frac{n\pi x}{a} \quad (3)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{a}\cos\frac{n\pi ct}{a}\sin\frac{n\pi x}{a}$$

then applying the last condition

$$\sum_{n=1}^{\infty} b_n \frac{n\pi c}{a} \sin\frac{n\pi x}{a} = V_0 \sin\left(\frac{\pi x}{a}\right)$$

$$b_1 \frac{\pi c}{a} \sin\frac{\pi x}{a} + b_2 \frac{2\pi c}{a} \sin\frac{2\pi c}{a} + \cdots$$

$$= V_0 \sin\left(\frac{\pi x}{a}\right)$$

$$b_1 \frac{\pi c}{a} = V_0 . \implies b_1 = \frac{V_0 a}{\pi c}.$$

$\therefore$ the solution is

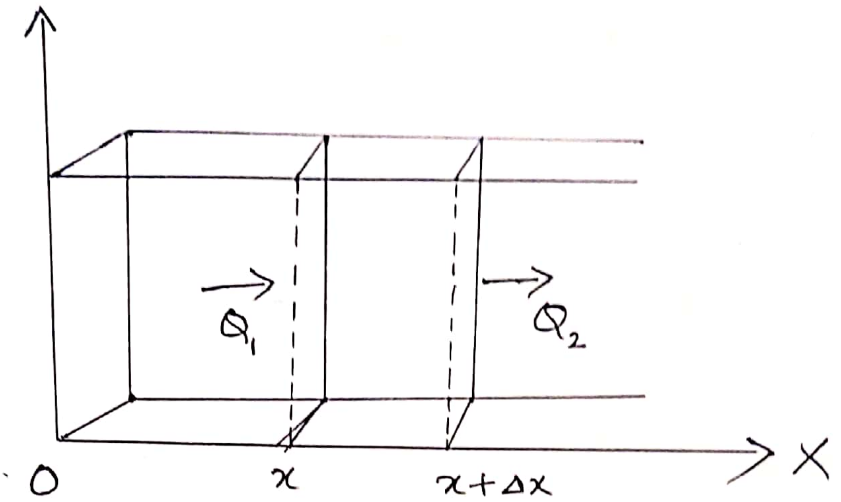
$$u(x,t) = b_1 \sin\frac{\cancel{n}\pi ct}{a} \sin\frac{\cancel{n}\pi x}{a}.$$

$$u(x,t) = \frac{V_0 a}{\pi c} \sin\frac{\cancel{n}\pi ct}{a} \sin\frac{\cancel{n}\pi x}{a}$$

6 A tightly streched string of length 'a' with fixed ends ~~is~~ initially in equilibrium position Find displacement $u(x,t)$ of the string if it is set vibrating by giving each of its points a velocity $V_0 \sin^3\left(\frac{\pi x}{a}\right)$?

ONE DIMENSIONAL HEAT EQUATION

DERIVATION



Consider the flow of heat by conduction in a uniform bar. It is assumed that the sides of the bar are insulated. and the loss of heat from the sides by conduction and radiation is negligible. Take one end of the bar as origin and the direction of flow of heat as the +ve X-axis. The temperature u at any point of the bar depends on the distance 'x' from one end and the time 't'. The amount of heat Q. crossing any section of the bar depends on the area of cross section 'A'. and the temperature gradient $\frac{\partial u}{\partial x}$ [ie the rate of change of temperature with respect to distance.

Let Q_1 be the amount of heat flowing into the section at a distance x

$$Q_1 \propto A\left(\frac{\partial u}{\partial x}\right)_x$$

ie, $$Q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x$$

Let Q_2 be the amount of heat flowing out of the section at a distance $x+\Delta x$.

$$Q_2 \propto A\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

$$Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

[-ve sign indicates that as x increases, u decreases]

The constant k is the thermal conductivity of the material. The amount of heat retained by the section is

$$Q_1 - Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_x + kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

$$Q_1 - Q_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \quad \text{—①}$$

The rate of increase of heat in the section

$$\propto \rho A \Delta x \, \cancel{\sigma} \, \frac{\partial u}{\partial t}$$

$$= \rho A \Delta x \, \sigma \, \partial u/\partial t \quad \text{—②}$$

where σ is the specific heat capacity

& ρ is the density of the material

$\frac{\partial u}{\partial t} \rightarrow$ temperature difference

From eqn (1) & (2) we have.

$$\left[\begin{array}{l} mc\theta \\ \rho = \frac{m}{V} = \frac{m}{A\Delta x} \\ m = \rho A \Delta x. \\ c = \sigma \\ \theta = \frac{\partial u}{\partial t} \end{array} \right.$$

$$kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] = \rho A \Delta x \sigma \frac{\partial u}{\partial t}$$

$$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} = \frac{\rho\sigma}{k}\frac{\partial u}{\partial t}$$

As $\Delta x \rightarrow 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho\sigma}{k}\frac{\partial u}{\partial t}$$

$$\frac{k}{\rho\sigma}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow c^2 u_{xx} = u_t$$

$$\cancel{\partial} = \frac{\rho\sigma}{\cancel{A}} = \cancel{A}$$

$c^2 = h = \frac{k}{\rho\sigma}$ is the thermal diffusivity of the material.

SOLUTIONS OF HEAT EQUATION BY THE METHOD OF SEPARATION OF VARIABLES

One dimensional heat equation is

$$c^2 u_{xx} = u_t \qquad u_{xx} = \frac{1}{h} u_t$$

$$h u_{xx} = u_t$$

$$u_{xx} = \frac{1}{h} u_t$$

Let $U = XT$ be the solution

$$U_x = X'T, \quad U_{xx} = X''T$$

$$U_t = XT'$$

Substituting we get

$$X''T = \frac{1}{h} XT'$$

$$\frac{X''}{X} = \frac{1}{h}\frac{T'}{T}$$

Let $\frac{X''}{X} = k$

$$X'' = kX$$

$$X'' - kX = 0$$

$$(D^2 - k)X = 0$$

$\therefore$ the auxiliary eqn is

$$m^2 - k = 0$$

$$m^2 = k$$

$$m = \pm\sqrt{k}$$

Let $\frac{1}{h}\frac{T'}{T} = k$

$$T' = khT$$

$$T' - khT = 0$$

$$(D - kh)T = 0$$

$\therefore$ the auxiliary eqn is

$$m - kh = 0$$

$$m = kh.$$

OR

$$\frac{T'}{T} = kh \Rightarrow \frac{dT}{T} = kh\,dt$$

$$\log T = kht + C \Rightarrow T = c_3 e^{kht}$$

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Case-I when $k=0$.

$m = 0, 0$

$X = (C_1 + C_2 x) e^{0x}$

$X = C_1 + C_2 x$

$m = 0$.

$T = C_3 e^{0t}$

$T = C_3$.

Case-II when k is +ve, say $k = p^2$

$\cancel{m = 0,0 \pm p}$

$m = \pm p$.

$X = C_1 e^{px} + C_2 e^{-px}$

$m = p^2 h$

$T = C_3 e^{p^2 ht}$.

Case-III when k is -ve say $k = -p^2$.

$m = \pm pi$

$X = e^{0x} [C_1 \cos px + C_2 \sin px]$

$X = C_1 \cos px + C_2 \sin px$.

$\cancel{m = -p^2 h}$

$m = -p^2 h$

$T = e^{-p^2 ht}$

$T = C_3 e^{-p^2 ht}$

$\therefore$ the solution is

$U = (C_1 + C_2 x) C_3$

$U = (C_1 e^{px} + C_2 e^{-px}) C_3 e^{p^2 ht}$

$U = [C_1 \cos px + C_2 \sin px] C_3 e^{-p^2 ht}$

But we need a solution in such a way that the temperature decreases as t increases.

$\therefore$ the only possible solution is

$$\boxed{u(x,t) = [c_1 \cos px + c_2 \sin px]\, c_3 e^{-p^2 h t}}$$

A long insulated rod with ends at zero temperature

Consider a rod of length l with given initial temperature distribution along its axis and ends at zero temperature. In this case the boundary conditions are $\boxed{u(0,t) = 0 \text{ and } u(l,t) = 0}$

Initial condition is $\boxed{u(x,0) = f(x)}$

1. Find the temperature distribution in a rod of length l, where end points are fixed at zero temp: and the initial temperature is $f(x)$.

1-D heat eqn is

$$u_{xx} = \frac{1}{h} u_t$$

By method of separation of variables the solution is given by

$$u(x,t) = [c_1 \cos px + c_2 \sin px]\, c_3 e^{-p^2 h t} \quad \text{—} (1)$$

B.C are $u(0,t) = 0$ & $u(l,t) = 0$

I.C are $u(x,0) = f(x)$

Applying the boundary condition $u(0,t)=0$, we get

$$C_1 C_3 e^{-p^2ht} = 0 \implies C_1 = 0$$

$$\therefore\ u(x,t) = C_2 C_3 \sin px\, e^{-p^2ht}$$

ie $u(x,t) = B \sin px\, e^{-p^2ht}$ where $B = C_2 C_3$ → (2)

Applying the boundary condition. $u(l,t)=0$. we get

$$B \sin pl \cdot e^{-p^2ht} = 0$$

$$\implies \sin pl = 0 \implies pl = n\pi$$

$$\cancel{\implies} \cancel{\sin pl} \implies p = \frac{n\pi}{l}$$

$$\therefore\ u(x,t) = B \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2ht}{l^2}}$$

Adding up the solutions we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2ht}{l^2}}$$

~~The~~ initial condition is $u(x,0) = f(x)$

Applying the initial condition we get

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = f(x)$$ which is a half range fourier sine series.

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$.

$\therefore$ the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2ht}{l^2}}$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

2 Find the temperature distribution in a rod of length 2m whose end points are maintained at temperature zero and the initial temperature $f(x) = 100(2x - x^2)$

The 1-D heat equation is

$$u_{xx} = \frac{1}{h} u_t$$

By method of separation of variables the solution is given by

$$u(x,t) = [C_1 \cos px + C_2 \sin px] C_3 e^{-p^2ht} \quad \text{—— (1)}$$

B.C are:

$$u(0,t) = 0 \quad \& \quad u(l,t) = 0 \text{ ie } u(2,t) = 0$$

I.C

$$u(x,0) = f(x) = 100(2x - x^2)$$

Applying the boundary condition $u(0,t) = 0$

we get $C_1 = 0$.

$$u(x,t) = C_2 C_3 \sin px \; e^{-p^2ht} \longrightarrow (2)$$

$$= B_n \sin px \; e^{-p^2ht} \longrightarrow (2)$$

Applying the 2nd boundary condition $u(l,t) = 0$

ie $u(2,t) = 0$

$$p = \frac{n\pi}{2}$$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2}\, e^{-\frac{n^2\pi^2}{4}ht} \qquad \text{--- (3)}$$

Applying the initial condition $u(x,0) = 100(2x - x^2)$

we get

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 100(2x - x^2)\sin\frac{n\pi x}{2}\,dx$$

$$= 100\left\{\left[(2x-x^2)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (2-2x)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right.\right.$$

$$\left.+ (-2)\left(\frac{\cos\frac{n\pi x}{2}}{\frac{n^3\pi^3}{2^3}}\right)\right]_0^2$$

$$= 100\left\{\left[0 - 0 - \frac{2\cdot 2^3}{n^3\pi^3}\cos n\pi\right] - \left[0 - 0 - \frac{2\cdot 2^3}{n^3\pi^3}\right]\right\}$$

$$= \frac{100\times 16}{n^3\pi^3}\left[1 - (-1)^n\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{3200}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ the solution is

$$u(x,t) = \sum_{n=1,3,5\ldots} \frac{3200}{n^3\pi^3} \sin\frac{n\pi x}{2}\, e^{-\frac{n^2\pi^2 ht}{4}}$$

3. Find the temperature in a laterally insulated bar of length L whose ends are kept at zero temperature if the initial temperature

$$f(x) = \begin{cases} x & 0 < x < \frac{L}{2} \\ L-x & \frac{L}{2} < x < L \end{cases}$$

The 1-D heat equation is

$$u_{xx} = \frac{1}{h} u_t$$

By method of separation of variables the solution is given by

$$u(x,t) = [c_1 \cos px + c_2 \sin px]\, c_3\, e^{-p^2 ht} \quad \text{—— (1)}$$

The boundary conditions are

$$u(0,t) = 0 \ \& \ u(L,t) = 0$$

Applying the boundary conditions $u(0,t)=0$

we have $c_1 = 0$.

Applying the 2nd condition $u(L,t)=0$.

we have $p = \frac{n\pi}{L}$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2 ht}{L^2}}$$

I.C are.

$$u(x,0) = f(x) = \begin{cases} x & 0 < x < \frac{L}{2} \\ L-x & \frac{L}{2} < x < L \end{cases}$$

Applying the initial condition. we get

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx.$$

$$= \frac{2}{L}\left\{ \int_0^{L/2} f(x) \sin\frac{n\pi x}{L}\, dx + \int_{L/2}^{L} f(x) \sin\frac{n\pi x}{L}\, dx \right\}$$

$$= \frac{2}{L}\left\{ \int_0^{L/2} x \sin\frac{n\pi x}{L}\, dx + \int_{L/2}^{L} (L-x) \sin\frac{n\pi x}{L}\, dx \right\}$$

$$= \frac{2}{L}\left\{ \left[(x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right) \right]_0^{L/2} + \right.$$

$$\left. \left[(L-x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right) \right]_{L/2}^{L} \right\}$$

$$= \frac{2}{L}\left\{ \left(\frac{L}{2}\right)\left(\frac{-\cos\frac{n\pi}{2}}{\frac{n\pi}{L}}\right) + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} - (0+0) \right.$$

$$+(0-0)-\left[\frac{-L}{2}\cdot\frac{L}{n\pi}\cos\frac{n\pi}{2}-\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\Bigg\}.$$

$$=\frac{2}{L}\left\{-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right.$$

$$=\frac{2}{L}\cdot 2\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}=\frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore\ u(x,t)=\sum_{n=1}^{\infty}\frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\,e^{-\frac{n^2\pi^2 ht}{L^2}}$$

4. Solve $\frac{\partial u}{\partial t}=h\frac{\partial^2 u}{\partial x^2}$ subject to the conditions

$u(0,t)=u(l,t)=0$ for $t>0$ and ~~$u(x,0)=3$~~

$u(x,0)=3\sin n\pi x \quad 0<x<\pi$

The 1-D heat equation is

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{h}\frac{\partial u}{\partial t}$$

By the method of separation of variables the solution is given by

$$u(x,t)=[C_1\cos px+C_2\sin px]\,C_3e^{-p^2ht}$$

Applying the boundary condition $u(0,t)=0$ we get

$C_1=0$

Applying the boundary condition $u(l,t)=0$ we have $p=n\pi/l$

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$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2 h t}{L^2}}$$

Applying the initial condition $L = 1$

$u(x,0) = 3 \sin n\pi x$ we get

$$\sum_{n=1}^{\infty} b_n \sin n\pi x = 3 \sin n\pi x$$

$$b_1 \sin \pi x + b_2 \sin 2\pi x + \cdots = 3 \sin n\pi x.$$

Comparing we get $b_1 = b_2 = \cdots = 3.$

i.e. $b_n = 3$ for all n.

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} 3 \sin n\pi x \ e^{-n^2\pi^2 h t}$$